# MATHEMATICAL SCIENCE

**Subject Code – 4** **Booklet Code – A**

**2013 (I)**

# TEST BOOKLET

**(23 June 2013)**

**Time Allowed: Three Hours** **Maximum Marks: 200**

## INSTRUCTIONS

1. **You have opted for English as medium of Question Paper.** This Test Booklet contains one hundred and twenty (20 Part 'A' + 40 Part 'B' + 60 Part 'C') Multiple Choice Questions (MCQs). You are required to answer a maximum of 15, 25 and 20 questions from part 'A', 'B' and 'C' respectively. If more than required number of questions are answered, only first 15, 25 and 20 questions in Part 'A', 'B' and 'C' respectively, will be taken up for evaluation.
2. Answer sheet has been provided separately. Before you start filling up your particulars, please ensure that the booklet contains requisite number of pages and that these are not torn or mutilated. If it is so, you may request the Invigilator to change the booklet. Likewise, check the answer sheet also. Sheets for rough work have been appended to the test booklet.
3. Write your Roll No., Name, Your address and Serial Number and this Test Booklet on the Answer sheet in the space provided on the side 1 of Answer sheet. Also put your signatures in the space identified.
4. You must darken the appropriate circles related to Roll Number, Subject Code, Booklet Code and Centre Code on the OMR answer sheet. It is the sole responsibility of the candidate to meticulously follow the instructions given on the Answer Sheet, failing which, the computer shall not be able to decipher the correct details which may ultimately result in loss, including rejection of the OMR answer sheet.
5. Each question in Part 'A' carries 2 marks, Part 'B', 3 marks and Part 'C' 4.75 marks respectively. There will be negative marking @0.5 marks in Part 'A' and @0.75 in Part 'B' for each wrong answer and no negative marking for part 'C.
6. Below each question in Part 'A' and 'B', four alternatives or responses are given. Only one of these alternatives is the "correct" option to the question. You have to find, for each question, the correct or the best answer. In Part 'C' each question may have **'ONE'** or **'MORE'** correct options. Credit in a question shall be given only on identification of **'ALL'** the correct options in Part 'C'. No credit shall be allowed in a question if any incorrect option is marked as correct answer.
7. Candidates found copying or resorting to any unfair means are liable to be disqualified from this and future examinations.
8. Candidate should not write anything anywhere except on answer sheet or sheets for rough work.
9. After the test is over, you MUST hand over the answer sheet (OMR) to the invigilator.
10. Use of calculator is not permitted.

Roll No. ………………………….

Name …………………………….

I have verified all the information

filled in by the candidate.

…………………………………………

Signature of the Invigilator

# MATHEMATICAL SCIENCE

## PART-A

(1.) What is the angle between the minute and hour hands of a clock at 7 : 35?

(a.) 0°

(b.) 17.5°

(c.) 19.5°

(d.) 20°

(2.) A stream of ants go from point A to point B and return to A along the same path. All the ants move at a constant speed and from any given point 2 ants pass per second one way. It takes 1 minute for an ant to go from A to B. How many returning ants will an ant meet in its journey from A to B?

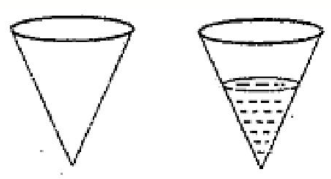
(a.) 120

(b.) 60

(c.) 240

(d.) 180

(3.)



The capacity of the conical vessel shown above is V. It is filled with water upto half its height. The volume of water in the vessel is

(a.) $\frac{V}{2}$

(b.) $\frac{V}{4}$

(c.) $\frac{V}{8}$

(d.) $\frac{V}{16}$

(4.) A large tank filled with water is to be emptied by removing half of the water present in it everyday. After how many days will there be closest to 10% water left in the tank?

(a.) One

(b.) Two

(c.) Three

(d.) Four

(5.) n is a natural number. If $n^5$ is odd, which of the following is true?

A. n is odd

B. $n^3$ is odd

C. $n^4$ is even

(a.) A only

(b.) B only

(c.) C only

(d.) A and B only

(6.) Suppose you expand the product $(x_1+y_1)(x_2+y_2)\ldots(x_{20}+y_{20})$. How many terms will have only one $x$ and rest $y's$ ?

(a.) 1

(b.) 5

(c.) 10

(d.) 20

(7.) A 16.2 m long wooden log has a uniform diameter of 2 m. To what length of log should be cut to obtain a piece of 22 $m^3$ volume?

(a.) 3.5 m

(b.) 7.0 m

(c.) 14.0 m

(d.) 22.0 m

(8.) What is the last digit of $7^{73}$?

(a.) 7

(b.) 9

(c.) 3

(d.) 1

(9.) A lucky man finds 6 pots of gold coins. He counts the coins in the first four pots to be 60, 30, 20 and 15, respectively. If there is a definite progression, what would be the numbers of coins in the next two pots?

(a.) 10 and 5

(b.) 4 and 2

(c.) 15 and 15

(d.) 12 and 10

(10.) A bee leaves its hive in the morning and after flying for 30 minutes due south reaches a garden and spends 5 minutes collecting honey. Then it files for 40 minutes due west and collects honey in another garden for 10 minutes. Then it returns to the hive taking the shortest route. How long was the bee away from its hive? (Assume that the bee flies at constant speed)

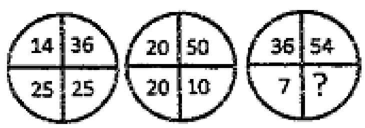
(a.) 85 min

(b.) 155 min

(c.) 135 min

(d.) Less than 1 hour

(11.) Find the missing number:




(a.) 1

(b.) 0

(c.) 2

(d.) 3

(12.) In solving a quadratic equation of the form $x^2 + ax + b = 0$, one student took the wrong value of $a$ and got the roots as 6 and 2; while another student took the wrong value of $b$ and got the roots as 6 and 1. What are the correct values of $a$ and $b$, respectively?

(a.) 7 and 12

(b.) 3 and 4

(c.) –7 and 12

(d.) 8 and 12

(13.) The distance between two oil rigs is 6 km. What will be the distance between these rings in maps of 1 : 50000 and 1 : 5000 scales, respectively?

(a.) 12 cm and 1.2 cm

(b.) 2 cm and 12 cm

(c.) 120 cm and 12 cm

(d.) 12 cm and 120 cm

(14.) A bird perched at the of a 12 m high tree sees a centipede moving towards the base of the tree from a distance equal to twice the height of the tree. The bird files along a straight line to catch the centipede. If both move at the same speed, at what distance from the base of the tree will the centipede be picked up by the bird?

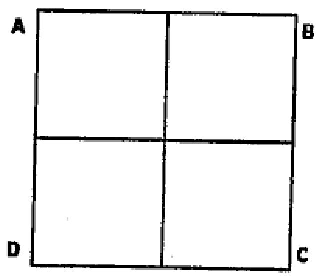
(a.) 16 m

(b.) 9 m

(c.) 12 m

(d.) 14 m

(15.) An ant goes from A to C in the figure crawling only on the lines and taking the least length of path. The number of ways in which it can do so is



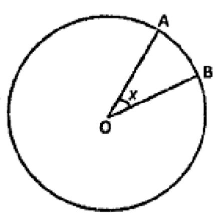

(a.) 2

(b.) 4

(c.) 5

(d.) 6

(16.) A point is chosen at random from a circular disc shown below. What is the probability that the point lies in the sector OAB?




(Where $\angle AOB = x$ radians)

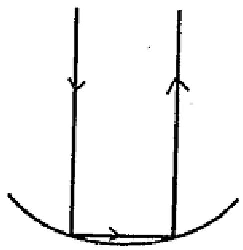
(a.) $\frac{2x}{\pi}$

(b.) $\frac{x}{\pi}$

(c.) $\frac{x}{2\pi}$

(d.) $\frac{x}{4\pi}$

(17.)



A ray of light, after getting reflected twice from a hemispherical mirror of radius R (see the above figure), emerges parallel to the incident ray. The separation of the original incident ray and the final reflected ray is

(a.) $R$

(b.) $R\sqrt{2}$

(c.) $2R$

(d.) $R\sqrt{3}$

(18.) A king ordered that a golden crown be made for him from 8 kg of gold and 2 kg of silver. The goldsmith took away some amount of gold and replaced it by an equal amount of silver and the crown when made, weighed 10 kg. Archimedes knew that under water gold lost 1/20$^{th}$ of its weight, while silver lost of1/10$^{th}$. When the crown was weighed under water, it was 9.25 kg. How much gold was stolen by the goldsmith?

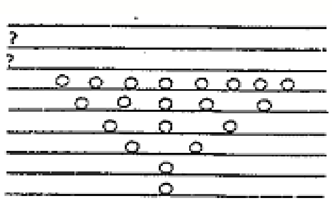
(a.) 0.5 kg

(b.) 1 kg

(c.) 2 kg

(d.) 3 kg

(19.) In the figure below the numbers of circles in the blank row must be

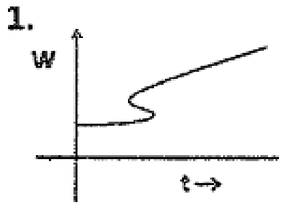


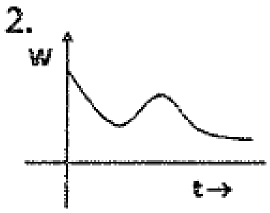
(a.) 12 and 20

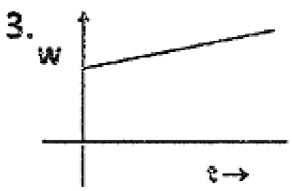
(b.) 13 and 20

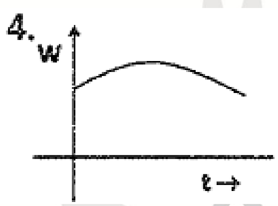
(c.) 13 and 21

(d.) 10 and 11

(20.) If we plot the weight (w) versus get (t) of a child in a graph, the one that will never be obtained from amongst the four graphs given below is (a.) 


(b.) 


(c.) 


(d.) 


## PART-B

(21.) Let $G$ be a simple group of order 168. What is the number of subgroups of $G$ of order 7?

(a.) 1

(b.) 7

(c.) 8

(d.) 28

(22.) The solution of the Cauchy problem for the first order PDE $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$, on $D=\{(x,y,z)\,|\,x^2+y^2\neq 0, z>0\}$, with the initial condition $x^2+y^2=1, z=1$ is

(a.) $z=x^2+y^2$

(b.) $z=\left(x^2+y^2\right)^2$

(c.) $z=\left(2-\left(x^2+y^2\right)\right)^{1/2}$

(d.) $z = \left(x^2 + y^2\right)^{1/2}$

(23.) The partial differential equation $\frac{\partial^2 u}{\partial y^2} - y\frac{\partial^2 u}{\partial x^2} = 0$ has

(a.) Two families or real characteristics curve for $y < 0$

(b.) No real characteristics for $y > 0$

(c.) Vertical lines as a family for characteristics curve for $y = 0$

(d.) Branches of quadratic curves as characteristics for $y \neq 0$

(24.) Consider the functional

$$I(y) = \int_a^b F(y, y')dx; \;\; y' \equiv \frac{dy}{dx}$$

$$y(a) = y_1, \quad y(b) = y_2$$

Where $y \in C^2[a, b]$, $F$ has order continuous partial derivatives with respect to $y, y'$, and $y_1, y_2$ are given real numbers. Let $y = y(x)$ be an extremizing function for the functional $I$. Then along the extremizing curve

(a.) $F$ remains constant

(b.) $\frac{\partial F}{\partial y} = 0$

(c.) $F - y\frac{\partial F}{\partial y'} =$ constant

(d.) $F - y'\frac{\partial F}{\partial y'} =$ constant

(25.) Consider the equation of an ideal planar pendulum given by

$$\frac{d^2 x}{dt^2} = -\sin x$$

Where $x$ denotes the angle of displacement. For sufficiently small angles of displacement, the solution is given by (where $a, b$ are constant)

(a.) $x(t) = a\cosh t + b\sinh t$

(b.) $x(t) = a + bt$

(c.) $x(t) = ae^t + be^{2t}$

(d.) $x(t) = a\cos t + b\sin t$

(26.) If the points $x_1, x_2, ..., x_n$ are distinct, then for arbitrary real values $y_1, y_2, ..., y_n$ the degree of the unique interpolating polynomial $p(x)$ such that $p(x_i) = y_i (1 \le i \le n)$ is

(a.) $n$

(b.) $n-1$

(c.) $\le n-1$

(d.) $\le n$

(27.) What is the smallest positive integer in the set $\{24x + 60y + 2000z \mid x, y, z \in \mathbb{Z}\}$ ?

(a.) 2

(b.) 4

(c.) 6

(d.) 24

(28.) Suppose observations on the pair $(X, Y)$ are:

| | | | | | | |
|---|---|---|---|---|---|---|
| $X$ | 1 | 7 | 5 | 9 | 11 | 3 |
| $Y$ | 20 | 68 | 58 | 70 | 181 | 37 |

Let $r_p$ and $r_s$ respectively denote the Pearson's and Spearman's rank correlation coefficient between $X$ and $Y$ based on the above data. Then which of the following is true?

(a.) $r_p = 1, r_s = 1$

(b.) $0 < r_p < 1, r_s = 1$

(c.) $r_p = 1, 0 < r_s < 1$

(d.) $0 < r_p < 1, 0 < r_s < 1$

(29.) Let, $f, g$ and $h$ be bounded functions on the closed interval $[a, b]$, such that $f(x) \le g(x) \le h(x)$ for all $x \in [a, b]$. Let $P = \{a = a_0 < a_1 < a_2 < ... < a_n = b\}$ be a partition of $[a, b]$. We denote by $U(f, P)$ and $L(f, P)$, the upper and lower Riemann sums of $f$ with respect to the partition $P$ and similarly for $g$ and $h$. Which of the following statements is necessarily true?

(a.) If $U(h, P) - U(f, P) < 1$ then $U(g, P) - L(g, P) < 1$

(b.) If $L(h, P) - L(f, P) < 1$ then $U(g, P) - L(g, P) < 1$

(c.) If $U(h, P) - L(f, P) < 1$ then $U(g, P) - L(g, P) < 1$

(d.) If $L(h, P) - U(f, P) < 1$ then $U(g, P) - L(g, P) < 1$

(30.) In a hypothesis-testing problem, which of the following is **NOT** required in order to compute the p-value?

(a.) Value of the test statistic

(b.) Distribution of the test statics under the null hypothesis

(c.) The level of significance

(d.) Whether the test is one-sided or two-sided

(31.) Which of the following series is convergent?

(a.) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}-\sqrt{n}}$

(b.) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

(c.) $\sum_{n=1}^{\infty} (-1)^n \log n$

(d.) $\sum_{n=1}^{\infty} \frac{\log n}{n}$



(32.) The general solution of the differential equation $\frac{d^2y}{dx^2}+y=f(x), x\in(-\infty, \infty)$, where $f$ is a continuous, real-valued function on $(-\infty, \infty)$, is (where $A, B, C$ and $k$ are arbitrary constants)



(a.) $y(x)=A\cos x+B\sin x+\int_0^x f(t)\sin(x-t)\,dt$

(b.) $y(x)=\cos(x+k)+C\int_0^x f(t)\sin(x-t)\,dt$

(c.) $y(x)=A\cos x+B\sin x+\int_0^x f(x-t)\sin t\,dt$

(d.) $y(x)=A\cos x+B\sin x+\int_0^x f(x+t)\cos t\,dt$

(33.) Consider the initial value problem (IVP)

$\frac{dy}{dx}=y^2, y(0)=1, (x, y)\in\mathbb{R}\times\mathbb{R}$.

Then there exists a unique solution of the IVP on

(a.) $(-\infty, \infty)$

(b.) $(-\infty, 1)$

(c.) $(-2, 2)$

(d.) $(-1, \infty)$

(34.) The power series $\sum_{n=0}^{\infty} \frac{\left[2+(-1)^n\right]^n}{3^n} x^n$ converges

(a.) Only for $x=0$

(b.) For all $x \in \mathbb{R}$

(c.) Only for $-1 < x < 1$

(d.) Only for $-1 < x \leq 1$

(35.) In which of the following cases, there is no continuous function $f$ from the set $S$ **onto** the set7?

(a.) $S=[0,\ 1], T=\mathbb{R}$

(b.) $S=(0,\ 1), T=\mathbb{R}$

(c.) $S=(0,\ 1), T=(0,\ 1]$

(d.) $S=\mathbb{R}, T=(0,\ 1)$

(36.) The function $f(x)=a_0+a_1|x|+a_2|x|^2+a_3|x|^3$ is differentiable at $x=0$

(a.) For no values of $a_0, a_1, a_2, a_3$

(b.) For any value of $a_0, a_1, a_2, a_3$

(c.) Only if $a_1=0$

(d.) Only if both $a_1=0$ and $a_3=0$.

(37.) Let $A=\begin{bmatrix} 1 & 3 & 5 & a & 13 \\ 0 & 1 & 7 & 9 & b \\ 0 & 0 & 1 & 11 & 15 \end{bmatrix}$ where $a, b \in \mathbb{R}$. Choose the correct statement.

(a.) There exist value of $a$ and $b$ for which the columns of $A$ are linearly independent.

(b.) There exist values of $a$ and $b$ for which $Ax=0$ has $x=0$ as the only solution.

(c.) For all values of $a$ and $b$ the rows of $A$ span a 3-dimensional subspace of $\mathbb{R}^5$

(d.) There exist values of $a$ and $b$ for which rank $(A)=2$.

(38.) Which of the following rings is a PID?

(a.) $\mathbb{Q}[X,Y]/(X)$

(b.) $\mathbb{Z}\oplus\mathbb{Z}$

(c.) $\mathbb{Z}[X]$

(d.) $M_2[\mathbb{Z}]$, the ring of 2 × 2 matrices with entries in $\mathbb{Z}$

(39.) Let $F \subseteq \mathbb{C}$ be the splitting field of $x^7 - 2$ over $\mathbb{Q}$, and $z = e^{2\pi i/7}$, a primitive seventh root of unity. Let $[F : \mathbb{Q}(z)] = a$ and $[F : \mathbb{Q}(\sqrt[7]{2})] = b$ .Then

(a.) $a = b = 7$

(b.) $a = b = 6$

(c.) $a > b$

(d.) $a < b$

(40.) Suppose $X_1$ and $X_2$ are independent and identically distributed random variables each following an exponential distribution with mean $\theta$, i.e., the common pdf is given by $f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty, 0 < \theta < \infty$ .

Then which of the following is true?

Conditional distribution of $X_2$ given $X_1 + X_2 = t$ is

(a.) Exponential with mean $\frac{t}{2}$ and hence $X_1 + X_2$ is sufficient for $\theta$

(b.) Exponential with mean $\frac{t\theta}{2}$ and hence $X_1 + X_2$ is not sufficient for $\theta$

(c.) Uniform $(0,\ t)$ and hence $X_1 + X_2$ is sufficient for $\theta$

(d.) Uniform $(0,\ t0)$ and hence $X_1 + X_2$ is not sufficient for $\theta$

(41.) A simple random sample of size $n$ is drawn the replacement (SRSWR) from a population of $N$ units. The expected number of distinct units in the sample is

(a.) $n\left[1 - \left(\frac{N-1}{N}\right)^n\right]$

(b.) $n\left[1 - \left(\frac{N-2}{N}\right)^n\right]$

(c.) $N\left[1 - \left(\frac{N-1}{N}\right)^n\right]$

(d.) $N\left[1 - \frac{n(N-1)^n}{N}\right]$

(42.) Consider a parallel system with two components. The lifetimes of the two components are independent and identically distributed random variables each following an exponential distribution with mean 1. The expected lifetime of the system is

(a.) 1

(b.) 1/2

(c.) 3/2

(d.) 2

(43.) The number of limit points of the set $\left\{\frac{1}{m}+\frac{1}{n}: m, n \in \mathbb{N}\right\}$ is

(a.) 1

(b.) 2

(c.) Finitely many

(d.) Infinitely many

(44.) Area enclosed between $x$-axis from $a$ to $b$ and the curve $f(x)$ is finite when

(a.) $a=0,\ b=\infty,\ f(x)=e^{-5x^5}$

(b.) $a=-\infty,\ b=\infty,\ f(x)=e^{-5x^5}$

(c.) $a=-7,\ b=\infty,\ f(x)=\frac{1}{x^4}$

(d.) $a=-7,\ b=7,\ f(x)=\frac{1}{x^4}$



(45.) Consider the following linear programming problem:

Maximize $z=3x_1+2x_2$

Subject to

1. $x_1+x_2 \geq 1$
2. $x_1+x_2 \leq 5$
3. $2x_1-3x_2 \leq 6$
4. $-2x_1+3x_2 \leq 6$

The problem has

(a.) An unbounded solution

(b.) Exactly one optimal solution

(c.) More than one optimal solution

(d.) No feasible solutions

(46.) Let $a, b, c$ be distinct real numbers. Then the number of distinct real roots of the equation $(x-a)^3+(x-b)^3+(x-c)^3=0$ is (a.) 1

(b.) 2

(c.) 3

(d.) Depends on the values of $a, b, c$

(47.) Consider the following subsets of $\mathbb{R}^2$, where $a, b \in \mathbb{R}$

$$A = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a \neq b\right\}$$

$$B = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, a \neq b\right\}$$

$$C = \{(x, y) \in \mathbb{R}^2 : ax + by + 5 = 0\}$$

$$D = \{(x, y) \in \mathbb{R}^2 : ax = by^2\}$$

$$E = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 = 1\}$$



Then which of the following is correct?

(a.) $C$ and $D$ are compact, but $A, B, E$ are not compact

(b.) $A$ and $B$ are compact, but $C, D, E$ are not compact

(c.) $A, B$ and $E$ are compact, but $C, D$ are not compact

(d.) $A$ and $E$ are compact, but $B, C, D$ are not compact.



(48.) Let $p(z), q(z)$ be two non-zero complex polynomials. Then $p(z)\overline{q(z)}$ is analytic if and only if

(a.) $p(z)$ is constant

(b.) $p(z)q(z)$ is constant

(c.) $q(z)$ is a constant

(d.) $\overline{p(z)}q(z)$ is a constant

(49.) If $z_1$ and $z_2$ are distinct complex numbers such that $|z_1| = |z_2| = 1$ and $z_1 + z_2 = 1$, then the triangle in the complex plane with $z_1, z_2$ and –1 as vertices

(a.) Must be equilateral

(b.) Must be right-angled

(c.) Must be isosceles, but not necessarily equilateral

(d.) Must be obtuse angled

(50.) Consider the following three populations in $\mathbb{R}^2$:

A. $A_1$ = {(−2, 0), (2, 0), (0, −2), (0, 2), (−1, −1), (−1, 1), (1, −1), (1, 1)}

B. $A_2$ = {(0, 0), (−1, 1), (1, 1), (−2, 3), (2, 3), (−3, 6), (3, 61)}

C. $A_3$ = {(−3, 0), (−2, 0), (0, 0), (1, 0), (2, 0), (3, 1)}

Suppose one point is selected at random from each population, the point from population $A_i$ being labeled $(X_i, Y_i), i = 1, 2, 3$. Then the absolute value of Cov $(X_i, Y_i)$ will be highest for

(a.) $i = 1$

(b.) $i = 2$

(c.) $i = 3$

(d.) $i = 2$ and $i = 3$

(51.) Let $J_1$ be the smallest topology on $\mathbb{R}^2$ containing the sets

$(a, b) \times (c, d)$ for all $a, b, c, d \in \mathbb{R}$;

$J_2$ be the smallest topology containing the sets

$\{(x, y) : (x-a)^2 + (y-b)^2 < c\}$ for all $a, b \in \mathbb{R},\ c > 0;$

$J_3$ be the smallest topology containing the sets

$\{(x, y) : |x-a| + |y-b| < c\}$ for all $a, b \in \mathbb{R}, c > 0$.

Which of the following is true?

(a.) $J_1 = J_2 = J_3$

(b.) $J_1 \not\subseteq J_2 \subseteq J_3$

(c.) $J_2 \not\subseteq J_3 \subseteq J_1$

(d.) $J_3 \not\subseteq J_2 \subseteq J_1$

(52.) Consider the following linear model

$Y_i = a + (-1)^i b + e_1;\ i = 1, \dots, n,\ \ n \geq 3$

Where $e_i's$ are independent and identically distributed random variables following normal distribution with mean zero and variance $\sigma^2$. Which of the following statements is correct?

(a.) The maximum likelihood estimators of $a$ and $b$ always exist

(b.) The maximum likelihood estimators of $a$ and $b$ always exist, but they may not be unique

(c.) The maximum likelihood estimator of $\sigma^2$ does not exist (d.) The maximum likelihood estimators of $a$ and $b$ are not consistent

(53.) Consider $\mathbb{R}^3$ with the standard inner product. Let $W$ be the subspace of $\mathbb{R}^3$ spanned by $(1, 0, -1)$. Which of the following is a basis for the orthogonal complement of $W$ ?

(a.) $\{(1, 0, 1), (0, 1, 0)\}$

(b.) $\{(1, 2, 1), (0, 1, 1)\}$

(c.) $\{(2, 1, 2), (4, 2, 4)\}$

(d.) $\{(2, -1, 2), (1, 3, 1), (-1, -1, -1)\}$

(54.) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables each having a uniform distribution on $(0,\ 1)$. Consider the histogram of these values with $k$ equally spaced class intervals given by $\{(a_i, b_i], i = 1, 2, \ldots, k\}$ where $a_i = \frac{i-1}{k}$, and $b_i = a_i + \frac{1}{k}$. Let $N_i$ be the number of values in the interval $(a_i, b_i]$.Then the covariance of $N_1$ and $N_k$ is

(a.) 0

(b.) $-n/k^2$

(c.) $n/k^2$

(d.) $1/2$

(55.) A linear transformation $T$ rotates each vector in $\mathbb{R}^2$ clockwise through 90°. The matrix $T$ relative to the standard ordered basis $\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$ is

(a.) $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

(b.) $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

(c.) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

(d.) $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

(56.) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Which of the following statements implies that $T$ is bijective?

(a.) Nullity $(T) = n$

(b.) Rank $(T)$ = Nullity $(T) = n$

(c.) Rank $(T)$ + Nullity $(T) = n$

(d.) Rank $(T)$ – Nullity $(T) = n$

(57.) $(X, Y)$ follows the bivariate normal distribution $N_2(0, 0, 1, 1, \rho), -1 < \rho < 1$. Then,

(a.) $X + Y$ and $X - Y$ are uncorrelated only if $\rho = 0$

(b.) $X + Y$ and $X - Y$ are uncorrelated only if $\rho < 0$

(c.) $X + Y$ and $X - Y$ are uncorrelated only if $\rho > 0$

(d.) $X + Y$ and $X - Y$ are uncorrelated for all values of $\rho$

(58.) The integral equation

$$y(x) = x - \int_1^x xy(t)\,dt; \quad y \in C^1(1, \infty]$$

has the solution

(a.) $y = x(1 - \ln x)$

(b.) $y = xe^{x - \frac{1}{2}}(x-1) + x$

(c.) $y = xe^{(1-x^2)^2}$

(d.) $y = x - (e^{x^2} - e)$

(59.) Let $U_1, U_2, \ldots$ be independent and identically distributed random each having a uniform distribution on $(0, 1)$. Then $\lim_{n\to\infty} P\left(U_1 + \ldots + U_n \le \frac{3}{4}n\right)$

(a.) Does not exist

(b.) Exists and equals 0

(c.) Exists and equals 1

(d.) Exists and equals $\frac{3}{4}$

(60.) Consider the quadratic equation $x^2 + 2U_x + V = 0$ where $U$ and $V$ are chosen independently and randomly from $\{1, 2, 3\}$ with equal probabilities. Then the probability that the equal has both roots real equals

(a.) $\frac{2}{3}$

(b.) $\frac{1}{2}$

(c.) $\frac{7}{9}$

(d.) $\frac{1}{3}$

## PART-C

(61.) Let $p$ be a real polynomial of the real variable $x$ of the form $p(x)=x^n+a_{n-1}x^{n-1}+...+a_1x-1$. Suppose that $p$ has no roots in the open unit disc and $p(-1)=0$. Then

(a.) $p(1)=0$

(b.) $\lim_{x\to\infty} p(x)=\infty$

(c.) $p(2)>0$

(d.) $p(3)=0$



(62.) Let $f$ be a function of $\mathbb{R}^2$ such that $\frac{\partial f}{\partial x}(x,y)=\frac{\partial f}{\partial y}(x,y)$ for all $(x,y)\in\mathbb{R}^2$.

(a.) $f(x,y)-f(x,y)=(x-y)\frac{\partial f}{\partial x}(x^*,y^*)\ +(y-x)\frac{\partial f}{\partial y}(x^*,y^*)$ for some point $(x^*,y^*)\in\mathbb{R}^2$



(b.) $f$ is a constant on all lines parallel to the line $x=-y$

(c.) $f(x,y)=0$ for all $(x,y)\in\mathbb{R}^2$

(d.) $f(x,y)=f(-y,x)$ for all $(x,y)\in\mathbb{R}^2$

(63.) Suppose $f:\mathbb{R}\to\mathbb{R}$ is a function that satisfies $|f(x)-f(y)|\le|x-y|^\beta$, $\beta>0$. Which of the following is correct?

(a.) If $\beta=1$ then $f$ is differentiable

(b.) If $\beta>0$ then $f$ is uniformly continuous

(c.) If $\beta>1$ then $f$ is a constant function

(d.) $f$ must be a polynomial

(64.) Let $S$ denote the set of all primes $p$ such that the following matrix is invertible when considered as a matrix with entries in $\mathbb{Z}/p\mathbb{Z}$.

$$A=\begin{pmatrix}1 & 2 & 0\\ 0 & 3 & -1\\ -2 & 0 & 2\end{pmatrix}$$

Which of the following statements are true?

(a.) $S$ contains all the prime numbers

(b.) $S$ contains all the prime number greater than 10

(c.) $S$ contains all the prime number other than 2 and 5

(d.) $S$ contains all the odd prime numbers.

(65.) Let $A \in M_{10}(\mathbb{C})$, the vector space of $10 \times 10$ matrices with entries in $\mathbb{C}$. Let $W_A$ be the subspace of $M_{10}(\mathbb{C})$ spanned by $\{A^n \mid n \geq 0\}$. Choose the correct statements

(a.) For any $A$, dim $(W_A) \leq 10$

(b.) For any $A$, dim $(W_A) < 10$

(c.) For some $A$, $10 < \dim(W_A) < 100$

(d.) For some $A$, $\dim(W_A) = 100$

(66.) Let $A$ be a complex 3 × 3 matrix with $A^3 = -1$. Which of the following statements are correct?

(a.) $A$ has three distinct eigenvalues

(b.) $A$ is diagonalizable over $\mathbb{C}$

(c.) $A$ is triangularizable over $\mathbb{C}$

(d.) $A$ is non-singular

(67.) Consider the quadratic forms $q$ and $p$ given by

$q(x, y, z, w) = x^2 + y^2 + z^2 + bw^2$ and

$p(x, y, z, w) = x^2 + y^2 + cwz$.

Which of the following statements are true?

(a.) $p$ and $q$ are equivalent over $\mathbb{C}$ if $b$ and $c$ are non-zero complex numbers.

(b.) $p$ and $q$ are equivalent over $\mathbb{R}$ if $b$ and $c$ are non-zero real numbers.

(c.) $p$ and $q$ are equivalent over $\mathbb{R}$ if $b$ and $c$ are non-zero real numbers with $b$ negative.

(d.) $p$ and $q$ are NOT equivalent over $\mathbb{R}$ if $c = 0$

(68.) Let $X_1, .., X_n$ be independent and identically distributed random variables with probability density function $f(x) = \frac{1}{2}\lambda^3 x^2 e^{-\lambda x}; x > 0,\ \lambda > 0$

Then which of the following statements are true?

(a.) $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is an unbiased estimator of $\lambda$

(b.) $\frac{3n}{\sum_{i=1}^{n} X_i}$ is an unbiased estimator of $\lambda$

(c.) $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is a consistent estimator of $\lambda$

(d.) $\frac{3n}{\sum_{i=1}^{n} X_i}$ is a consistent estimator of $\lambda$

(69.) Consider a Markov chain on the state space $\{1, 2, 3, 4, 5\}$ with transition probability matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 & \frac{2}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

Then which of the following statements are true?

(a.) States 1, 2, 4 are recurrent and states 3, 5 are transient

(b.) States 1, 2, 3, 4 are recurrent and state 5 is transient

(c.) The chain has a unique stationary distribution

(d.) The chain has more than one stationary distributions

(70.) Let $X$ and $Y$ be independent random variables each following uniform distribution on $(0, 1)$. Let $W = X\, I_{[Y \le X^2]}$, where $I_A$ denotes the indicator function of the set $A$. Then which of the following statements are true?

(a.) The cumulative distribution function of $W$ is given by $F_W(t) = t^2 I_{\{0\le t\le 1\}} + I_{\{t>1\}}$

(b.) $P[W > 0] = \frac{1}{3}$

(c.) The cumulative distribution function of $W$ is continuous

(d.) The cumulative distribution function of $W$ is given by $F_W(t) = \left(\frac{2+t^3}{3}\right) I_{\{0\le t\le 1\}} + I_{\{t>1\}}$

(71.) Let $X$ be a random variable with probability density function

$$f(x) = \alpha (x-\mu)^{\alpha-1} e^{-(x-\mu)^{\alpha}};\ -\infty < \mu < \infty,\ \alpha > 0,\ x > \mu\ .$$

The hazard function is

(a.) Constant for all $\alpha$

(b.) An increasing function for some $\alpha$

(c.) Independent of $\alpha$

(d.) Independent of $\mu$ when $\alpha = 1$

(72.) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables each following a uniform distribution on $(0,\ 1)$. Denote $T_n = \max\{X_1, X_2, \ldots, X_n\}$. Then, which of the following statements are true?

(a.) $T_n$ converges to 1 in probability

(b.) $n(1-T_n)$ converges in distribution

(c.) $n^2(1-T_n)$ converges in distribution

(d.) $\sqrt{n}(1-T_n)$ converges to 0 in probability.

(73.) Consider the following optimization problem:

Maximize $3x+4y+2z,$

Subject to $x+y+z \le 12$

$x+2y-z \le 5$

$x-y+z \le 2$

Where $x, y, z \ge 0$. Then

(a.) The problem has more than one feasible solution

(b.) The objective function of the dual problem is to minimize $12u+5v+2w$

(c.) One of the constraints of the dual problem is $u-v+w \ge 2$

(d.) Two of the constraints of the dual problem are $u+v+w \le 3, u+2v-w \le 4$

(74.) Let $X_1, X_2, \ldots$ be independent random variables each following a normal distribution with unknown mean $\mu$ and unknown variance $\sigma^2 > 0$. Define

$$\bar{X}_{n-2} = \frac{1}{n-2}\sum_{i=1}^{n-1} X_i,\ T_1 = \frac{\sum_{i=1}^{n-2}\left(X_i - \bar{X}_{n-2}\right)^2}{n-3} \text{ and } T_2 = \frac{\left(X_{n-1} - X_n\right)}{\sqrt{2}};\ n > 3.$$

Then which of the following statements are correct?

(a.) $T_1$ is unbiased for $\sigma^2$

(b.) $\frac{T_2}{\sqrt{T_1}}$ follows a $t$ distribution with $(n-3)$ degree of freedom (c.) $\frac{T_2^2}{T_1}$ follows a $F$ distribution with 1 and $(n-3)$ degrees of freedom

(d.) $\bar{X}_{n-2}$ is consistent for estimating $\mu$

(75.) Let $X$ be a non-negative integer valued random variable with probability mass function $f(x)$ satisfying $(x+1)f(x+1)=(\alpha+\beta x)f(x), x=0,1,2,...,\beta\neq 1$. You may assume that $E(X)$ and $\mathrm{Var}(X)$ exist. Then which of the following statements are true?

(a.) $E(X)=\frac{\alpha}{1-\beta}$

(b.) $E(X)=\frac{\alpha^3}{(1-\beta)(1+\alpha)}$

(c.) $Var(X)=\frac{\alpha^2}{(1-\beta)^2}$

(d.) $Var(X)=\frac{\alpha}{(1-\beta)^2}$

(76.) Consider the model

$Y_{ijk}=\mu+\alpha_i+\beta_{ij}+\varepsilon_{ijk};\ i,j,k=1,2,...,5$

Where $\varepsilon_{ijk}$ are independent and identically distributed random variables each following a normal distribution with mean 0 and variance $\sigma^2>0$, and $\mu,\alpha_i,\beta_{ij},i,j=1,2,...,5$ are fixed parameters. Then which of the following statements are true?

(a.) $\mu$ is estimable

(b.) All linear function of $\alpha_i, i=1,2,...,5$ are estimable

(c.) $\mu+\alpha_1+\beta_{12}$ is estimable

(d.) $\beta_{21}-\beta_{22}$ is estimable.

(77.) Let $X_1, X_2, ...., X_8$ be a random sample from the normal distribution with mean $\theta$ and variance 1, and let the prior distribution of $\theta$ be normal with mean 2 and variance 2. Define $\bar{X}=\frac{1}{8}\sum_{i=1}^{8}X_i$ .

Then which of the following statements are true?

(a.) The prior is a conjugate prior

(b.) Posterior mean of $\theta$ given $\bar{X}$ is $\frac{16\bar{X}+2}{17}$

(c.) For absolute error loss, the Bayes estimator is $\frac{16\bar{X}+2}{17}$

(d.) For squared error loss, the Bayes estimator is $\frac{16\overline{X}+2}{17}$

(78.) Let $X$ be a random variable with density function

$$f(x)=\begin{cases}\frac{2\theta x+1}{\theta+1}, & 0\le x\le 1,\ \theta>-1\\ 0\quad, & \text{otherwise}\end{cases}$$

Consider the problem of testing $H_0:\theta\le 1$ against $H_1:\theta>1$

Let $\phi$ be the test given by

$$\phi(x)=\begin{cases}1 & \text{if } x\ge \frac{\sqrt{9-8\alpha}-1}{2}\\ 0 & \text{if } x<\frac{\sqrt{9-8\alpha}-1}{2}\end{cases}$$

Then which of the following statements are true?



(a.) $\phi$ is a UMP size $\alpha$ test

(b.) $\phi$ is not a UMP size $\alpha$ test

(c.) For all $\theta>1$, the power of the test $\phi$ at $\theta$ is at least $\alpha$

(d.) For some $\theta>1$ the power of the test $\phi$ at $\theta$ can be less than $\alpha$

(79.) Consider a $2^5$ factorial experiment laid out as a block design with 4 blocks of size 8 each. Suppose the principal block of this design consists of the treatment combinations (1), ab, de and five others. Which of the following interaction effects can be confounded in this design?



(a.) ABC, CDE, ABDE

(b.) ABC, CDE, ABCDE

(c.) AB, BC, AC

(d.) AB, CDE, ABCDE

(80.) For a bivariate data set $(x_i, y_i), i=1,2,\dots,n$ suppose the least squares regression lines are:

Equation 1: $5x-8y+14=0$

Equation 2: $2x-5y+11=0$

Then which of the following statements are true?

(a.) The value of the correlation coefficient is –0.80

(b.) The value of the correlation coefficient is 0.80

(c.) The standard deviation of $y$ is less than the standard deviation of $x$

(d.) $(\bar{x}, \bar{y}) = (2, 3)$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

(81.) Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables each following a uniform distribution on $(\theta - 2, \theta + 2)$. Define $X_{(n)} = \max\{X_1, X_2, ..., X_n\}$ and $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$. Then which of the following estimators are maximum likelihood estimators for $\theta$?

(a.) $X_{(1)} - 2$

(b.) $X_{(n)} + 2$

(c.) $\frac{X_{(1)} + X_{(n)}}{2}$

(d.) $\frac{1}{4}(X_{(1)} + 2) + \frac{3}{4}(X_{(n)} - 2)$

(82.) Consider a design with 4 treatments labeled 1, 2, 3, 4 and with 5 blocks given by

$\boxed{1, \ 2}$ $\boxed{1, \ 3}$ $\boxed{1, \ 2, \ 4}$ $\boxed{1, \ 2, \ 3}$ $\boxed{1, \ 2, \ 3, \ 4}$

Which of the following statements are true?

(a.) The design is connected but not orthogonal

(b.) The design is connected and orthogonal

(c.) All treatment contrasts are estimable

(d.) Only some pairwise treatment contrasts are estimable

(83.) A simple random sample of size $n$ is to be drawn from a large population to estimate the population proportion $\theta$. Let $p$ be the sample proportion. Using the normal approximation, determine which of the following sample size values will ensure $|p - \theta| \le 0.02$ with probability at least 0.95, irrespective of the true value of $\theta$? [You may assume $\Phi(1.96) = 0.75$, $\Phi(1.64) = 0.95$, where $\Phi$ denotes the cumulative distribution function of the standard normal distribution].

(a.) $n = 1000$

(b.) $n = 1500$

(c.) $n = 2500$

(d.) $n = 3000$

(84.) Let $X_1, X_2, X_3, X_4, X_5$ be independent and identically distributed random variables each following a uniform distribution non $(0, 1)$, and let $M$ denote their median. Then which of the following statements are true?

(a.) $P\left(M < \frac{1}{3}\right) = P\left(M > \frac{2}{3}\right)$

(b.) $M$ is uniformly distributed on $(0,\ 1)$

(c.) $E(M) = E(X_1)$

(d.) $V(M) = V(X_1)$

(85.) A linear operator $T$ on a complex vector space $V$ has characteristics polynomial $x^3(x-5)^2$ and minimal polynomial $x^2(x-5)$. Choose all correct options.

(a.) The Jordan form of $T$ is uniquely determined by the given information

(b.) There are exactly 2 Jordan blocks in the Jordan decomposition of $T$

(c.) The operator induced by $T$ on the quotient space $V/\mathrm{Ker}(T-5I)$ is nilpotent, where $I$ is the identity operator

(d.) The operator induced by $T$ on the quotient space $V/\mathrm{Ker}(T)$ is a scalar multiple of the identity operator

(86.) Consider the function $f(z) = z^2(1-\cos z),\ z \in \mathbb{C}$. Which of the following are correct?

(a.) The function $f$ has zeros of order 2 at 0

(b.) The function $f$ has zeros of order 1 at $2\pi n,\ \ n = \pm 1,\ \pm 2, \ldots$

(c.) The function $f$ has zeros of order 4 at 0

(d.) The $f$ has zeros of order 2 at $2\pi n,\ n = \pm 1, \pm 2, \ldots$

(87.) Let $B$ be an open subset of $\mathbb{C}$ and $\partial B$ denote the boundary of $B$. Which of the following statements are correct?

(a.) For every entire function $f$, we have $\partial(f(B)) \subseteq f(\partial B)$

(b.) For every entire function $f$ and a bounded open set $B$, we have $\partial(f(B)) \subseteq f(\partial B)$

(c.) For every entire function $f$, we have $\partial(f(B)) = f(\partial B)$

(d.) There exist an unbounded open subset $B$ of $\mathbb{C}$ and an entire function $f$ such that $\partial(f(B)) \subseteq f(\partial B)$

(88.) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Which of the following are correct?

(a.) There exists a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ and $f'(0) = 2$

(b.) There exists a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f\left(\frac{3}{4}\right) = \frac{3}{4}$ and $f'\left(\frac{2}{3}\right) = \frac{3}{4}$

(c.) There exist a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f\left(\frac{3}{4}\right) = \frac{-3}{4}$ and $f'\left(\frac{3}{4}\right) = \frac{-3}{4}$

(d.) There exists a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f\left(\frac{1}{2}\right) = \frac{-1}{2}$ and $f'\left(\frac{1}{4}\right) = 1$

# MATHEMATICAL SCIENCE

(89.) Let $f:\mathbb{C}\to\mathbb{C}$ be a analytic function. For $z=x+iy$, let $u,v:\mathbb{R}^2\to\mathbb{R}$ be such that $u(x,y)=\operatorname{Re} f(z)$ and $v(x,y)=\operatorname{Im} f(z)$. Which of the following are correct?

(a.) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

(b.) $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

(c.) $\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y\partial x}=0$

(d.) $\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 v}{\partial y\partial x}=0$

(90.) Let $\sigma=(1\ 2)\ (3\ 4\ 5)$ and $\tau=(1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$ be permutations in $S_6$, the group of permutations on six symbols. Which of the following statements are true?

(a.) The subgroups $\langle\sigma\rangle$ and $\langle\tau\rangle$ are isomorphic to each other

(b.) $\sigma$ and $\tau$ are conjugate in $S_6$

(c.) $\langle\sigma\rangle\cap\langle\tau\rangle$ is the trivial group

(d.) $\sigma$ and $\tau$ commute

(91.) Let $S_n$ denote the symmetric group on $n$ symbols. The group $S_3\oplus(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to which of the following groups?

(a.) $\mathbb{Z}/12\mathbb{Z}$

(b.) $(\mathbb{Z}/6\mathbb{Z})\oplus(\mathbb{Z}/2\mathbb{Z})$

(c.) $A_4$, the alternating group of order 12

(d.) $D_6$, the dihedral group of order 12

(92.) Let $F=F_3[X]/(x^3+2x-1)$, where $F_3$ is the field with 3 elements. Which of the following statements are true?

(a.) $F$ is a field with 27 elements

(b.) $F$ is a separable but not a normal extension of $F_3$

(c.) The automorphism group of $F$ is cyclic

(d.) The automorphism group of $F$ is abelian but not cyclic

(93.) Which of the polynomials are irreducible over the given rings?

(a.) $X^5+3X^4+9X+15$ over $\mathbb{Q}$, the field of rationals

(b.) $X^3+2X^2+X+1$ over $\mathbb{Z}/7\mathbb{Z}$, the ring of integers modulo 7 (c.) $X^3+X^2+X+1$ over $\mathbb{Z}$, the ring of integers

(d.) $X^4+X^3+X^2+X+1$ over $\mathbb{Z}$, the ring of inters.

(94.) Consider the boundary value problem (BVP)

$u''=-f, u(0)=u''(1)=0$ on $[0,\ 1]$,

Where $u'\equiv\frac{du}{dx}\quad u''\equiv\frac{d^2u}{dx^2}$. Assume $f(x)$ is real-valued continued function on $[0,\ 1]$. Then, which of the following are correct?

(a.) The Green's function $G(x,\zeta),(x,\zeta)\in[0,\ 1]\times[0,\ 1]$, for the above BVP is

$$G(x,\zeta)=\begin{cases} x & \text{for } 0\le x\le\zeta \\ \zeta & \text{for } \zeta\le x\le 1\end{cases}$$

(b.) Both $G$ and $\frac{\partial G}{\partial x}$ are continuous on $[0,\ 1]\times[0,\ 1]$ with $\frac{\partial^2 G}{\partial x^2}$ having a discontinuity along $x=\zeta$

(c.) $G(x,\zeta)$ satisfies the homogenous equation $u''=0$ for $0\le x<\zeta$ and $\zeta<x\le 1$

(d.) The solution of the given BVP is $u(x)=\int_0^x \zeta f(\zeta)d\zeta+\int_x^1 xf(\zeta)d\zeta$

(95.) Consider the congruence $x^n\equiv 2$ (mod 13).This congruence has a solution for $x$ if

(a.) $n=5$

(b.) $n=6$

(c.) $n=7$

(d.) $n=8$

(96.) Consider the two sets $A=\{1,\ 2,\ 3\}$ and $B=\{1,\ 2,\ 3,\ 4,\ 5\}$. Choose the correct statements.

(a.) The total number of functions from $A$ to $B$ is 125

(b.) The total number of functions from $A$ to $B$ is 243

(c.) The total number of one-to-one functions from $A$ to $B$ is 60

(d.) The total number of one-to-one functions from $A$ to $B$ is 120

(97.) Consider $\mathbb{Q}$, the set of rational numbers, with the metric $d(p,\ q)=|p-q|$. Then which of the following are true?

(a.) $\{q\in\mathbb{Q}\,|\,2<q^2<3\}$ is closed

(b.) $\{q\in\mathbb{Q}\,|\,2\le q^2\le 4\}$ is compact

(c.) $\{q\in\mathbb{Q}\,|\,2\le q^2\le 4\}$ is closed (d.) $\{q \in \mathbb{Q} \mid q^2 \geq 1\}$ is compact

(98.) Which of the following define a metric on $\mathbb{R}$?

(a.) $d(x, y) = \dfrac{|x-y|}{1+|x-y|}$

(b.) $d(x, y) = |x-2y| + |2y-x|$

(c.) $d(x, y) = |x^2 - y^2|$

(d.) $d(x, y) = |x^3 - y^3|$

(99.) Let $y_{app}(x)$ be a polynomial approximation, involving only one coordinate function, for the functional

$$I(y) = \int_0^1 \left(\frac{1}{2}y'^2 - y\right)dx\,;\; y(0) = 0,\; y(1) = 0,$$

using Rayleigh-Ritz method; here $y \in C^2[0,\ 1]$. If $y_e(x)$ is an exact extremizing function, then $y_e$ and $y_{app}$ are coincident at

(a.) $x = 0$ but not at remaining point [0, 1]
(b.) $x = 1$ but not at remaining points in [0, 1]
(c.) $x = 0$ $x = 1$, and but not at other points in [0, 1]
(d.) All point $x \in [0,\ 1]$

(100.) Let $f : \mathbb{R} \to [0,\ \infty)$ be a non-negative real valued continuous function.

Let $\phi_n(x) = \begin{cases} n & \text{if } f(x) \geq n \\ 0 & \text{if } f(x) < n \end{cases}$, $\phi_{n,k}(x) = \begin{cases} \dfrac{k}{2^n} & \text{if } f(x) \in \left[\dfrac{k}{2^n}, \dfrac{k+1}{2^n}\right) \\ 0 & \text{if } f(x) \notin \left[\dfrac{k}{2^n}, \dfrac{k+1}{2^n}\right) \end{cases}$

And $g_n(x) = \phi_n(x) + \sum_{k=0}^{n2^n-1} \phi_{n,k}(x)$. As $n \uparrow \infty$, which of the following are true?

(a.) $g_n(x) \uparrow f(x)$ for every $x \in \mathbb{R}$

(b.) Given any $C > 0$, $g_n(x) \uparrow f(x)$ uniformly on the set $\{x : f(x) < C\}$

(c.) $g_n(x) \uparrow f(x)$ uniformly for $x \in \mathbb{R}$

(d.) Given any $C > 0$, $g_n(x) \uparrow f(x)$ uniformly on the set $\{x : f(x) \geq C\}$

(101.) Let $X_1, X_2, \ldots$ be independent random variables each following exponential distribution with mean 1. Then which of the following statements are correct?

(a.) $P(X_n > \log n$ for infinitely many $n \geq 1) = 1$

(b.) $P(X_n > 2\log n$ for infinitely many $n \geq 1) = 1$

(c.) $P(X_n > \frac{1}{2}\log n$ for infinitely many $n \geq 1) = 0$

(d.) $P(X_n > \log n, X_{n+1} > \log(n+1)$ infinitely many $n \geq 1) = 0$

(102.) Let $A_n \subseteq \mathbb{R}$ for $n \geq 1$, and $\chi_n : \mathbb{R} \to \{0, 1\}$ be the function

$$\chi_n(x) = \begin{cases} 0 & \text{if } x \notin A_n \\ 1 & \text{if } x \in A_n \end{cases}$$

Let $g(x) = \limsup_{n\to\infty} \chi_n(x)$ and $h(x) = \liminf_{n\to\infty} \chi_n(x)$

(a.) If $g(x) = h(x) = 1$, then there exists m such that for all $n \geq m$ we have $x \in A_n$

(b.) If $g(x) = 1$ and $h(x) = 0$, then there exist m such that for all $n \geq m$ we have $x \in A_n$

(c.) If $g(x) = 1$ and $h(x) = 0$ then there exists a sequence $n_1, n_2, \ldots$ of distinct integers such that $x \in A_{n_k}$ for all $k \geq 1$

(d.) If $g(x) = h(x) = 0$ then there exists m such that for all $n \geq m$ we have $x \notin A_n$.

(103.) Which of the following function $f$ is uniformly continuous on the interval (0, 1)?

(a.) $f(x) = \frac{1}{x}$

(b.) $f(x) = x \sin\frac{1}{x}$

(c.) $f(x) = \sin\frac{1}{x}$

(d.) $f(x) = \frac{\sin x}{x}$

(104.) The minimum possible value of $|z|^2 + |z-3|^2 + |z-6i|^2$, where $z$ is a complex number and $i = \sqrt{-1}$, is

(a.) 15

(b.) 45

(c.) 30

(d.) 20

(105.) Let $f : \mathbb{R}^n \to \mathbb{R}$ be the map $f(x_1, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n$, where $a = (a_1, \ldots, a_n)$ is a fixed non-zero vector. Let $Df(0)$ denote the derivative of $f$ at 0. Which of the following are correct?

(a.) $(Df)(0)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$

(b.) $[(Df)(0)](a) = \|a\|^2$

(c.) $[(Df)(0)](a)=0$

(d.) $[(Df)(0)](b)=a_1b_1+...+a_nb_n$ for $b=(b_1,...,b_n)$

(106.) Let $F_1, F_2:\mathbb{R}^2\to\mathbb{R}$ be the functions $F_1(x_1,x_2)=\frac{-x_2}{x_1^2+x_2^2}$ and $F_2(x_1,x_2)=\frac{x_1}{x_1^2+x_2^2}$. Which of the following are correct?

(a.) $\frac{\partial F_1}{\partial x_2}=\frac{\partial F_2}{\partial x_1}$

(b.) There exists a function $f:\mathbb{R}2\setminus\{(0,0)\}\to\mathbb{R}$ such that $\frac{\partial f}{\partial x_1}=F_1$ and $\frac{\partial f}{\partial x_2}=F_2$

(c.) There exists no function $f:\mathbb{R}2\setminus\{(0,0)\}\to\mathbb{R}$ such that $\frac{\partial f}{\partial x_1}=F_1$ and $\frac{\partial f}{\partial x_2}=F_2$

(d.) There exists a function $f:D\to\mathbb{R}$ where $D$ is the open disc of radius 1 centred at (2, 0), which satisfies $\frac{\partial f}{\partial x_1}=F_1$ and $\frac{\partial f}{\partial x_2}=F_2$ on $D$.

(107.) Let $A$ be a subset of $\mathbb{R}^p$ and $x\in\mathbb{R}^p$. Denote $d(x,A)=\inf\{d(x,y):y\in A\}$. There exists a point $y_0\in A$ with $d(y_0,x)=d(x,A)$, if

(a.) $A$ is any closed non-empty subset of $\mathbb{R}^p$

(b.) $A$ is any non-empty subset of $\mathbb{R}^p$

(c.) $A$ is any non-empty compact subset of $\mathbb{R}^p$

(d.) $A$ is any non-empty bounded subset of $\mathbb{R}^p$

(108.) Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function with period $\rho>0$. Then $g(x)=\int_x^{x+p} f(t)\,dt$ is a

(a.) Constant function

(b.) Continuous function

(c.) Continuous function but not differentiable

(d.) Neither continuous nor differentiable

(109.) Let $z=x+iy$ and $f:\mathbb{R}^2\to\mathbb{R}^2$ be the function $f(x,y)=f(z)=z^2=(x^2-y^2,2xy)\in\mathbb{R}^2$. Let $(Df)(a)$ denote the derivative of $f$ at $a$. Which of the following are true?

(a.) $(Df)(a)h=2\,a\,h$, where $a=a_1+ia_2$ and $h=h_1+ih_2$

(b.) $(Df)(a) = 2\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}, a = (a_1, a_2) \in \mathbb{R}^2$

(c.) $f$ is one to one on $\mathbb{R}^2$

(d.) For any $a \in \mathbb{R}^2 / \{(0,0)\}, f$ is one to one on some neighbourhood of $a$.

(110.) Let $A$ be the set of rational numbers in the open interval $(0,\ 7)$ and $f : A \to \mathbb{R}$ be a uniformly continuous function. Which of the following are true?

(a.) $f$ is bounded

(b.) $f$ is necessarily a constant function

(c.) $f$ is differentiable on (0, 7)

(d.) $f$ is differentiable at all the rational points in (0, 7)

(111.) Let the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$, $t \geq 0, \vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ admit an exponential function $\exp(i(\vec{k} \cdot \vec{x} + wt))$ as its solution, where $\vec{k}$ is a non zero constant real vector, and $w$ is constant. Then the solution

(a.) Remain constants on certain planes in $\mathbb{R}^3$

(b.) Repeats itself after a certain length $L$

(c.) Has, in general, an amplitude decaying exponentially with time $t$

(d.) Is bounded uniformly for $\vec{x} \in \mathbb{R}^3$ for a fixed $t$

(112.) Consider the Laplace equation in polar form:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0; \quad 0 < r \leq a, 0 \leq \theta < 2\pi$$

Satisfying $u(a, \theta) = f(\theta)$, where $f$ is a given function. Let $\sigma$ be the separation constant that appears when one uses the method of separation of variables. Then for solution $u(r, \theta)$ to be bounded and also periodic in $\theta$ with period $2\pi$,

(a.) $\sigma$ cannot be negative

(b.) $\sigma$ can be zero, and in that case the solution is a constant

(c.) $\sigma$ can be positive, and in the case it must be an integer

(d.) The fundamental set of solutions is $\{1, r^n \sin n\theta, r^n \cos n\theta\}$, where $n$ is a positive integer.

(113.) For the homogenous Fredholm equation

$$y(x) = \lambda \int_0^{\pi} \sin(x + \zeta)\, y(\zeta)\, d\zeta,$$

The eigenvalue $\lambda$ and the corresponding eigen function $y(x)$, involving arbitrary constants $A$ and $B$, are

(a.) $\lambda = \frac{2}{\pi},\ y(x) = A(\sin x - \cos x)$

(b.) $\lambda = \frac{-2}{\pi},\ y(x) = B(\sin x + \cos x)$

(c.) $\lambda = \frac{-2}{\pi},\ y(x) = B(\sin x - \cos x)$

(d.) $\lambda = \frac{2}{\pi},\ y(x) = A(\sin x + \cos x)$

(114.) Consider the motion of a rigid body around a stationary point 0. Let $M_1, M_2$ and $M_3$ be the components of the angular momentum vector along the three principal axes. Let $I_1, I_2$ and $I_3$ be the moments of inertia. Which of the following are conserved?

(a.) $M_1^2 I_1 + M_2^2 I_2 + M_3^2 I_3$

(b.) $\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}$

(c.) $M_1^2 + M_2^2 + M_3^2$

(d.) $M_1^2 I_1^2 + M_2^2 I_2^2 + M_3^2 I_3^2$

(115.) Consider a sufficiently smooth function $f(x)$. A formula for estimating its derivative is given by $\frac{df}{dx} = \frac{1}{4h}\left[f(x+2h) - f(x-2h)\right] +$ error term, where $h > 0$. Let $f^{(n)}$ denote the $n^{\text{th}}$ derivative of $f$ and let $\zeta$ be a point between $x-2h$ and $x+2h$. Which of the following expressions for the error term are correct?

(a.) $\frac{-f^{(2)}(\zeta)h^2}{2}$

(b.) $\frac{-2f^{(3)}(\zeta)h^2}{3}$

(c.) $-f^{(1)}(\zeta)h$

(d.) $\frac{-f^{(4)}(\zeta)h^4}{12}$

(116.) A Runge-Kutta method for numerically solving the initial-value ordinary differential equation $y' = f(x, y);\ y(x_0) = y_0$ is given by (for $h$ small)

$$y(x+h) = y(x) + W_1 F_1(x, y) + W_2 F_2(x, y)$$

$$F_1(x, y) = hf(x, y)$$

$F_2(x, y) = hf(x + \alpha h, y + \beta F_1)$.

The objective is to determine the constants $W_1, W_2, \alpha$ and $\beta$ such that the above formula is accurate to order 2 (there is, the error term is $O(h^3)$). Which of the following are correct sets of values for these constants ?

(a.) $W_1 = \frac{1}{2}, W_2 = \frac{1}{2}, \alpha = 1, \beta = 1$

(b.) $W_1 = 2, W_2 = 1, \alpha = \frac{1}{2}, \beta = \frac{1}{2}$

(c.) $W_1 = \frac{1}{3}, W_2 = \frac{2}{3}, \alpha = \frac{3}{4}, \beta = \frac{3}{4}$

(d.) $W_1 = \frac{3}{4}, W_2 = \frac{1}{4}, \alpha = 2, \beta = 2$

(117.) The extremal of $\int_1^2 \frac{\dot{x}^2}{t^3} dt; x(1) = 3,\ x(2) = 18$ (where $\dot{x} \equiv \frac{dx}{dt}$) using Lagrange's equation is given by which of the following?

(a.) $x = t^4 + 2$

(b.) $x = \frac{15}{7}t^3 + \frac{6}{7}$

(c.) $x = 5t^2 - 2$

(d.) $x = 5t^3 + 3$

(118.) Consider the first order PDE $p + q = pq$ where $p \equiv \frac{\partial z}{\partial x}, q \equiv \frac{\partial z}{\partial y}$. Then which of the following are correct?

(a.) The Charpit's equations for the above PDE reduce to $\frac{dx}{1-q} = \frac{dy}{1-p} = \frac{dz}{-pq} = \frac{dp}{p+q} = \frac{dq}{0}$

(b.) A solution of the Charpit's equation is $q = b,$ where $b$ is a constant.

(c.) The corresponding value of $p$ is $p = \frac{b}{b-1}$

(d.) A solution of the equation is $z = \frac{b}{b-1}x + by + a,$ where $a$ and $b$ are constants.

(119.) Consider the boundary value problem (BVP)

$$u'' + \lambda u = 0, u(0) = u'(\pi) = 0, u' \equiv \frac{du}{dx}, u'' \equiv \frac{d^2u}{dx^2}, \lambda \in \mathbb{C}$$

Let $k$ denote a non negative integer.

Then, which of the following are correct?

(a.) There exist eigenvalues of the BVP and the corresponding eigen functions constitute an orthogonal set.

(b.) The eigen values of the BVP are $\left(k+\frac{1}{2}\right)^2$ with the corresponding eigen functions $\left\{\sin\left(k+\frac{1}{2}\right)x\right\}$.

(c.) The eigenvalues of the BVP are $(k+1)^2$ with the corresponding eigen functions $\{\sin(k+1)x\}$.

(d.) There exists no non real eigen values for the BVP.

(120.) Consider the initial value problem (IVP) $\frac{dy}{dx}=xy^{1/3},\ y(0)=0, (x,\ y)\in\mathbb{R}\times\mathbb{R}$. Then, which of the following are correct?

(a.) The function $f(x,y)=xy^{1/3}$ does not satisfy a Lipschitz condition with respect to $y$ in any neighbourhood of $y=0$

(b.) There exists a unique solution for the IVP

(c.) There exists no solution for the IVP

(d.) There exist more than one solution for the IVP